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THE CORRESPONDENCE BETWEEN CUT-ELIMINATION AND NORMALIZATION II

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Table of Contents

* Part I, on intuitionistic predicate logic, appeared in the previous issue.

Part II. HEYTING'S ARITHMETIC

§8. The systems $\mathcal{S}$(H) and $\mathcal{N}$(H) *

8.1. *Introduction*

We will give two formulations of Heyting's arithmetic: in the sequent calculus and in natural deduction. These systems will be called $\mathcal{S}$(H) and $\mathcal{N}$(H) respectively, or just $\mathcal{S}$ and $\mathcal{N}$ again. Their negative fragments will be denoted by $\mathcal{S}^-$(H) and $\mathcal{N}^-$(H) (or just $\mathcal{S}^-$ and $\mathcal{N}^-$).

8.2. *Language*

The language includes:
individual variables (free and bound);
the individual constant 0;
constants for all primitive recursive functions, including successor;
the predicate constant =.

Terms are defined inductively as follows: variables and 0 are terms; and if f is an n-ary function constant and $t_1, ..., t_n$ are terms, then $ft_1 ... t_n$ is a term.

We write t^+ for the successor of t, $s+t$ for $+st$, etc. Numerals are defined as usual, and $\bar{n}$ denotes the numeral for the number n.

With each closed term t is associated a unique *value* $|t|$, according to the intended interpretation of 0 and the function constants.

Formulas. Atomic formulas have the form: $s = t$. (We do not now have the propositional constant $\perp$.) Formulas are built up from these as in §2.1.

* This section should be compared with §2.

Note. We will use the notation = both metamathematically, to denote syntactic identity, and for formal equality. This should not cause confusion.

8.3. *The system* $\mathcal{S}(\mathrm{H})$

8.3.1. *Symbols, indices, indexed formulas* and *sequents* are defined as in §2.2.

8.3.2. *Initial sequents.* There are two kinds:

(a) *Logical initial sequents* (LIS): $A_i \to A$ for any formula A and initial index i satisfying the restriction on indices (2.2.7(a)).

(b) *Mathematical initial sequents* (MIS) have the form

$$\begin{aligned} &\to P\,, \\ Q_i &\to P\,, \\ Q_i, R_j &\to P \end{aligned}$$

with P, Q, R atomic and initial indices i, j satisfying the restriction on indices. (So, in particular, $i \neq j$.) (We will usually omit these indices.) These sequents are of three kinds:

(i) *Equality axioms*:

$$\to t = t\ ;$$
$$s = t, \quad P(s) \to P(t)$$

for any terms s, t and atomic formula $P(a)$.

(ii) *Peano's axioms for* 0 *and successor*:

$$s^+ = t^+ \to s = t\ ;$$
$$t^+ = 0 \ \to r = s$$

for any terms r, s, t.

Note. An initial sequent of the form $t^+ = 0 \to t^+ = 0$ must be *labelled* as either a LIS or MIS (cf. the problem with $\bot \to \bot$, §5.13).

(iii) *Defining equations for all primitive recursive functions*, e.g.

$$\to s + 0 = s\ ;$$
$$\to s + t^+ = (s + t)^+$$

for all terms s, t; etc.

8.3.3. *Inference rules.* There are three kinds:

(a) *Structural,* and

(b) *Logical,* as in 2.2.6, and also

(c) *Induction* (Ind):

$$\frac{\Gamma \to F0 \quad (Fa_\alpha), \Delta \to Fa^+}{\Gamma, \Delta \to Ft}$$

for any formula Fa, with α possibly empty (cf. 2.2.8(b)), and the

Restriction on variables: $a \notin \Delta \cup \{F0\}$.

Here Ft is called the *induction formula*, t the *induction term* and a the *proper variable* of the inference.

Note. It is generally more natural to call Fa the "induction formula", but our definition will be more convenient for our purposes (see 8.3.5(vi)).

8.3.4. **Notation.** (a) Extending the convention of 2.2.8(b), we write

$$(Q), (R) \to P$$

to denote any MIS (with 0, 1 or 2 formulas on the left; i.e., Q and/or R may be missing).

(b) We write $\mathcal{I}$, $\mathcal{I}_1$, $\mathcal{I}'$, ... for derivations ending with Ind.

(c) For a derivation $\mathcal{D}$ consisting (only) of a MIS, we write: $r\mathcal{D} = \mathrm{M}$ (cf. 2.2.9(b)).

(d) Convention (C) (2.2.4) is still assumed. Again, in this connection, Propositions 1 and 2 of 2.2.12 hold.

8.3.5. **Definition** (cf. 2.2.10). A formula occurrence in a derivation is:

(i) *logical initial* (LI) if it is in a LIS,

(ii) *mathematical initial* (MI) if it is in a MIS,

(iii)–(v) *principal, contracted* or *passive* as in 2.2.10,

(vi) *induction* (Ind) if it is the induction formula of an Ind inference (i.e. on the right of the LS of such an inference).

Proposition. *Every formula occurrence in a derivation of* $\mathcal{S}(\mathrm{H})$ *is one and* *one only of the above; except that if it is on the left side of a sequent it cannot be induction, and if on the right, cannot be contracted.*

8.3.6. *Seven types of cut* (cf. 2.2.11). Now it is convenient to classify cuts into seven types (not mutually exclusive): (a)–(d) as in 2.2.11, and also:

(e) cut formula is MI in both US's,
(f) cut formula is Ind in LUS, and MI in RUS,
(g) cut formula is Ind in LUS, and principal in RUS.

Note. If cut formula is Ind in LUS and not MI or principal in RUS, then this cut falls into type (a), (b) or (c).

8.4. *The system* $\mathcal{N}$(H)

8.4.1. *Rules for constructing derivations in* $\mathcal{N}$(H). (a) *Trivial derivations*, (b) *contraction*, and (c) *logical inference* rules (except $\perp$): as in 2.3.7.

In addition there are two new kinds of inference:

(d) *Mathematical*: there is an "M-inference" corresponding to each MIS of $\mathcal{S}$(H), viz.:

for each MIS $\to P$ there is a 0-premiss rule or axiom P;

for each MIS $Q \to P$ there is a 1-premiss rule $\dfrac{Q}{P}$;

for each MIS $Q, R \to P$ there is a 2-premiss rule $\dfrac{Q, R}{P}$.

(These M-inferences form an "atomic system", as defined by Prawitz [17, II, 1.5].)

(e) *Induction* (Ind):

$$\dfrac{\begin{matrix}\Gamma & [Fa_\alpha], \Delta \\ \Pi_1 & \Pi_2 \\ F0 & Fa^+\end{matrix}}{Ft}$$

with the

Restriction on variables: $a \notin \Delta \cup \{F0\}$.

Here exactly one assumption class (Fa_α) is discharged (if $\alpha \neq \emptyset$), or none is (if $\alpha = \emptyset$) (cf. 2.3.4(a)).

$F0$ is called the *minor premiss*, Fa^+ the *major premiss*, Ft the *induction formula*, t the *induction term*, and a the *proper variable* of this inference; and the occurrences of a shown in Fa and Fa^+ are called its occurrences at this inference.

(There may be contractions between assumption classes in Γ and Δ.)

8.4.2. Notation. (a) We must distinguish between a (trivial) derivation of a formula A from itself, and one consisting of an axiom (i.e., 0-premiss M-inference) A. So we denote the latter by placing a bar above the A: $\overline{A}$.

So all three kinds of M-inference shown in 8.4.1 (d) can be represented by the schema

$$\frac{(Q), (R)}{P} .$$

(b) For a derivation Π ending with an M-inference or induction, we write (respectively) $r\Pi$ = M or Ind (cf. 2.3.9(b)).

8.5. *The map* φ : $\mathrm{Der}(\mathcal{S}(\mathrm{H})) \to \mathrm{Der}(\mathcal{N}(\mathrm{H}))$

8.5.1. Definition. This extends the map φ defined in § 2.4 (ignoring the case $r\mathcal{D} = \bot$), in an obvious way:

$r\mathcal{D}$	$\mathcal{D}$	$\varphi\mathcal{D}$
M	$\to P$	$\overline{P}$
	$Q_i \to P$	$\dfrac{Q_i}{P}$
	$Q_i, R_j \to P$	$\dfrac{Q_i, R_j}{P}$
Ind	$\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ \Gamma \to F0 & (Fa_\alpha), \Delta \to Fa^+\end{matrix}}{\Gamma, \Delta \to Ft}$	$\dfrac{\begin{matrix}\Gamma & [Fa_\alpha], \Delta\\ \varphi\mathcal{D}_1 & \varphi\mathcal{D}_2\\ F0 & Fa^+\end{matrix}}{Ft}$

8.5.2. Proposition. *φ is onto. Also its restriction to* Der($\mathcal{S}^-$(H)) *is onto* Der($\mathcal{N}^-$(H)).

Proof. As for 2.4.4, 2.4.6.

8.6. *The proper variable property in* $\mathcal{S}$(H) *and* $\mathcal{N}$(H)

The definitions and results remain as in § 2.5, except that we must add the phrase "or Ind" in 2.5.2 after "$\forall$R or $\exists$L", and also after "$\forall$I or $\exists$E". Also, add to 2.5.1 (a): "(vi) A_i is in the discharged class $[Fa_\alpha]$ and A_{i+1} is Fa^+ (in the Ind shown in 8.4.1(e)), or vice versa"; to 2.5.1(c): "and the proper variable of an Ind in Π does not occur in the Ind term"; and to 2.5.1(d): "and every occurrence of a proper variable of an Ind in $\mathcal{D}$ is in or above the RUS of that inference".

8.7. *Standard derivations*

8.7.1. Let t be a closed term with $|t| = n$. We can define a *standard derivation in* $\mathcal{N}$ *of* the formula $t = \bar{n}$. This contains only M-inferences, and gives essentially the computation of t. (Those M-inferences are used which correspond to the equality axioms or the defining equations for the primitive recursive functions used in constructing t. We omit details.)

8.7.2. Let Fa be any formula, and t a closed term with $|t| = n$. We define a *standard derivation in* $\mathcal{N}$ *of* $F\bar{n}$ *from* (an assumption class) Ft, and *of* Ft *from* (an assumption class) $F\bar{n}$. Both are defined simultaneously, by induction on the complexity of Fa, as follows.

(1) Fa is atomic. The derivations are

$$\frac{\begin{array}{c}\Pi\\ t=\bar{n}\end{array}\quad Ft}{F\bar{n}}\,(\mathrm{M}_1) \qquad \text{and} \qquad \frac{\dfrac{\begin{array}{c}\Pi\\ t=\bar{n}\end{array}\quad \dfrac{}{t=t}}{\bar{n}=t}\,(\mathrm{M}_2)\quad F\bar{n}}{Ft}\,(\mathrm{M}_3)$$

respectively, where M_1, M_2 and M_3 are M-inferences corresponding to the second equality axiom schema in $\mathcal{S}$ (8.3.2(b)(i)) (where, in M_2, $P(a)$ is $a = t$), and Π is a standard derivation of $t = \bar{n}$.

(2) Fa is $Ga \supset Ha$. The standard derivation of $F\bar{n}$ from Ft is

$$\dfrac{\dfrac{\begin{array}{c}[G\bar{n}_{\beta\times\alpha}]\\ \Pi_1\\ Gt\end{array} \qquad Gt \supset Ht_\alpha}{\begin{array}{c}(Ht_\alpha)\\ \Pi_2\\ H\bar{n}\end{array}}}{G\bar{n} \supset H\bar{n}}$$

where Π_1 and Π_2 are standard derivations which exist by induction hypothesis (for certain α, β).

Similarly for the standard derivation of Ft from $F\bar{n}$.

(3) Fa is $Ga \wedge Ha$. The standard derivation of $F\bar{n}$ from Ft is

$$\dfrac{\dfrac{Gt \wedge Ht_\alpha}{\begin{array}{c}(Gt_\alpha)\\ \Pi_1\\ G\bar{n}\end{array}} \qquad \dfrac{Gt \wedge Ht_\beta}{\begin{array}{c}(Ht_\beta)\\ \Pi_2\\ H\bar{n}\end{array}}}{G\bar{n} \wedge H\bar{n}}$$

where Π_1 and Π_2 are standard derivations which exist by induction hypothesis (for certain α, β).

Similarly for the standard derivation of Ft from $F\bar{n}$.

(4) Fa is $\forall x\, G(x, a)$. Similarly.

(5) Fa is $Ga \vee Ha$. The standard derivation of $F\bar{n}$ from Ft is

$$\dfrac{Gt \vee Ht_i \qquad \dfrac{\begin{array}{c}[Gt_\alpha]\\ \Pi_1\\ G\bar{n}\end{array}}{G\bar{n} \vee H\bar{n}} \qquad \dfrac{\begin{array}{c}[Ht_\beta]\\ \Pi_2\\ H\bar{n}\end{array}}{G\bar{n} \vee H\bar{n}}}{G\bar{n} \vee H\bar{n}}$$

where Π_1 and Π_2 are standard derivations which exist by induction hypothesis (for certain α, β).

Similarly for the standard derivation of Ft from $F\bar{n}$.

(6) Fa is $\exists x\, G(x, a)$. Similarly.

8.7.3. Now (again for any Fa, t closed, $|t| = n$) we can define a *standard derivation in $\mathcal{S}$ of $F\bar{n}_\beta \to Ft$* (for a suitable β) as some derivation $\mathcal{D}$ such that:

(i) $\varphi\mathcal{D}$ is the standard derivation in $\mathcal{N}$ of Ft from $F\bar{n}$, and

(ii) the only cuts in $\mathcal{D}$ are of atomic formulas and proper subformulas of Ft and $F\bar{n}$.

(Here Ga is considered as a subformula of $\forall x\, Gx$ and $\exists x\, Gx$, for any free variable a not in $\forall x\, Gx$.)

Note. $\mathcal{D}$ is defined in this way since it cannot really be chosen canonically. (We could, if we wanted, write down more conditions on $\mathcal{D}$; e.g., it should contain no logical initial sequents.)

8.7.4. **Notation.** Standard derivations are denoted by

$$\begin{array}{c} \mathcal{D}_{\mathrm{st}} \\ F\bar{n}_\beta \to Ft \end{array} \quad \text{and} \quad \begin{array}{c} F\bar{n}_\beta \\ \Pi_{\mathrm{st}} \\ Ft \end{array}$$

They will be used in the "induction conversions" (§9).

§9. Conversions in $\mathcal{S}^-$(H) and $\mathcal{N}^-$(H) *

9.1. *Conversions in* $\mathcal{S}^-$(H)

9.1.1. The *permutative conversions* are defined as in §3: permutative cut conversions as in 3.1.1 and contraction conversions as in 3.1.2.

Essential conversions however are now of two kinds:

(a) *logical conversions* (i.e. ⊃-, ∧- and ∀-convns) as in 3.1.3; and

(b) *induction conversions*: Let

$$\mathcal{D} = \cfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to F0\end{matrix} \qquad \begin{matrix}\mathcal{D}_2(a) \\ (Fa_\alpha), \Delta \to Fa^+\end{matrix}}{\Gamma, \Delta \to Ft} \quad ,$$

t closed, $|t| = n$. There are three cases:

(i) Ind-convn$_1$: $n = 0$.

$$\mathcal{D} \text{ conv } \cfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to F0\end{matrix} \qquad \begin{matrix}\mathcal{D}_{\text{st}} \\ F0_\beta \to Ft\end{matrix}}{\Gamma_{\times\beta} \to Ft} \text{ (Cut)} \qquad \text{(see 8.7.4).}$$

(ii) Ind-convn$_2$: $n > 0$, $\alpha = \emptyset$.

$$\mathcal{D} \text{ conv } \cfrac{\begin{matrix}\mathcal{D}_2(\overline{n-1}) \\ \Delta \to F\bar{n}\end{matrix} \qquad \begin{matrix}\mathcal{D}_{\text{st}} \\ F\bar{n}_\beta \to Ft\end{matrix}}{\Delta_{\times\beta} \to Ft} \text{ (Cut).}$$

(iii) Ind-convn$_3$: $n > 0$, $\alpha \neq \emptyset$. $\mathcal{D}$ conv

$$\cfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to F0\end{matrix} \quad \begin{matrix}\mathcal{D}_2(0) \\ F0_\alpha, \Delta \to F1\end{matrix} \quad \begin{matrix}\mathcal{D}_2(1) \\ F1_\alpha, \Delta \to F2\end{matrix} \quad \ldots \quad \begin{matrix}\mathcal{D}_2(\overline{n-1}) \\ F(\overline{n-1})_\alpha, \Delta \to F\bar{n},\end{matrix} \quad \begin{matrix}\mathcal{D}_{\text{st}} \\ F\bar{n}_\beta \to Ft\end{matrix}}{\cfrac{\vdots \qquad \qquad (n+1 \text{ cuts, and contrs})}{\Gamma_{\times\gamma}, \Delta_{\times\delta} \to Ft}}$$

* This section should be compared with §3.

where $\gamma = \alpha^n \times \beta$,

$\delta = \beta \cup (\alpha \times \beta) \cup (\alpha^2 \times \beta) \cup \ldots \cup (\alpha^{n-1} \times \beta)$,

$\alpha^i =_{\mathrm{df}} \alpha \times \ldots \times \alpha$ (i factors),

and the dots represent $n+1$ cuts (with cut formulas $F0, F1, \ldots, F\bar{n}$), and contractions of corresponding formulas in the n Δ's (all of this performed in any order).

Note. With the three new types of cut listed in 8.3.6, no conversion is applicable! We summarize this situation as follows.

9.1.2. Remark (cf. 3.1.4). (a) Given any *cut* in a derivation, at least one of the cut conversions (in 3.1.1 and 3.1.3) applies to it, except in the following three cases:

(i) cut formula is MI in both US's,

(ii) cut formula is Ind in LUS, and MI in RUS,

(iii) cut formula is Ind in LUS, and principal in RUS.

(b) Given any *contraction* followed by another inference, one of the contraction conversions (in 3.1.2) applies to it, *except* in the two cases:

$$\dfrac{\dfrac{\Gamma, A_\alpha, A_\beta \to B}{\Gamma, A_{\alpha\cup\beta} \to B}\,(\mathrm{Contr})}{\Gamma \to A \supset B}\,(\supset\mathrm{R}) \quad \text{and} \quad \dfrac{\Gamma \to F0 \qquad \dfrac{Fa_\alpha, Fa_\beta, \Delta \to Fa^+}{Fa_{\alpha\cup\beta}, \Delta \to Fa^+}\,(\mathrm{Contr})}{\Gamma, \Delta \to Ft}\,(\mathrm{Ind})$$

9.1.3. *Pruning.* We extend the definition in 3.1.5 with the new case:

(9) $r\mathcal{D} = \mathrm{Ind}$.

$$\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to F0\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ (Fa_\alpha), \Delta \to Fa^+\end{matrix}}{\Gamma, \Delta \to Ft}\,.$$

(a) $$\mathcal{D}_1 \;\mathrm{conv}\; \begin{matrix}\mathcal{D}'_1 \\ \Gamma' \to F0\end{matrix} \;\Rightarrow\; \mathcal{D} \;\mathrm{conv}\; \dfrac{\mathcal{D}'_1 \quad \mathcal{D}_2}{\Gamma', \Delta \to Ft}\,.$$

(b) $$\mathcal{D}_2 \;\mathrm{conv}\; \begin{matrix}\mathcal{D}'_2 \\ (Fa_{\alpha'}), \Delta' \to Fa^+\end{matrix} \;\Rightarrow\; \mathcal{D} \;\mathrm{conv}\; \dfrac{\mathcal{D}_1 \quad \mathcal{D}'_2}{\Gamma, \Delta' \to Ft}\,.$$

Note. It is possible here that $\alpha \neq \emptyset$ but $\alpha' = \emptyset$.

9.2. *Conversions in* $\mathcal{N}^-(\mathrm{H})$

(a) *Logical conversions*, i.e., removal of maximal formulas: as in §3.2.

(b) *Induction conversions.* Let

$$
\Pi = \begin{array}{c}
\begin{array}{cc} \Gamma & [Fa_\alpha], \Delta \\ \Pi_1 & \Pi_2(a) \\ F0 & Fa^+ \end{array} \\ \hline Ft, \quad \Theta \\ \Pi_3
\end{array}
$$

t closed, $|t| = n$.

(There may be contractions between assumption classes in two or more of Γ, Δ and Θ, and some of these may be discharged in Π_3. For simplicity of notation, these are not shown explicitly here, but it should be clear how to deal with them: cf. §3.2.)

As in $\mathcal{S}^-(\mathrm{H})$, there are three cases:

(i) Ind-convn$_1$: $n = 0$.

$$
\Pi \quad \text{conv} \quad \begin{array}{c} \Gamma_{\times\beta} \\ \Pi_1 \\ (F0_\beta) \\ \Pi_{\mathrm{st}} \\ (Ft), \Theta \\ \Pi_3 \end{array} \qquad \text{(see 8.7.4).}
$$

(ii) Ind-convn$_2$: $n > 0$, $\alpha = \emptyset$.

$$
\Pi \quad \text{conv} \quad \begin{array}{c} \Delta_{\times\beta} \\ \Pi_2(\overline{n-1}) \\ (F\bar{n}_\beta) \\ \Pi_{\mathrm{st}} \\ (Ft), \ \Theta \\ \Pi_3 \end{array} .
$$

(iii) Ind-convn$_3$: $n > 0$, $\alpha \neq \emptyset$.

$$
\Pi \quad \text{conv} \quad
\begin{array}{l}
\Gamma_{\times\alpha^n\times\beta} \\
\Pi_1 \\
(F0_{\alpha^n\times\beta}),\ \Delta_{\times\alpha^{n-1}\times\beta} \\
\Pi_2(0) \\
(F1_{\alpha^{n-1}\times\beta}),\ \Delta_{\times\alpha^{n-2}\times\beta} \\
\Pi_2(1) \\
\vdots \\
\Pi_2(\overline{n-2}) \\
(F(\overline{n-1})_{\alpha\times\beta}),\ \Delta_{\times\beta} \\
\Pi_2(\overline{n-1}) \\
(F\bar{n}_\beta) \\
\Pi_{st} \\
(Ft),\ \Theta \\
\quad \Pi_3
\end{array}
$$

9.3. *Remarks on the induction conversions in* $\mathcal{S}^-$(H) *and* $\mathcal{N}^-$(H)

9.3.1. Our Ind-conversion rules for $\mathcal{N}^-$(H) and $\mathcal{S}^-$(H) can be criticized because they involve inserting a (logically complex) standard derivation of Ft from $F\bar{n}$ (or of $F\bar{n} \to Ft$), which seems to change the meaning of the derivation (see [14]). (This can be seen particularly in the case that t is identical to $\bar{n}$.)

One way of avoiding this is as follows. Define two formulas, A and A', to be *computationally equivalent* if there is a formula $F(a_1, ..., a_n)$ and closed terms $s_1, ..., s_n, t_1, ..., t_n$, with $|s_i| = |t_i|$ for $1 \leqslant i \leqslant n$, such that A is $F(s_1, ..., s_n)$ and A' is $F(t_1, ..., t_n)$. Now add to $\mathcal{N}$ the inference rule

Comp: $\dfrac{A}{A'}$,

and add to $\mathcal{S}$ the initial sequents $A_i \to A'$ and the rules

$$\frac{\Gamma \to A}{\Gamma \to A'}\ \text{(Comp R)} \qquad \text{and} \qquad \frac{A_\alpha, \Gamma \to B}{A'_\alpha, \Gamma \to B}\ \text{(Comp L)}$$

where, in all cases, A is computationally equivalent to A'. (An approach like this was taken in e.g. [18].)

Then, for Ind-conversions, we use Comp inferences $\frac{F\bar{n}}{Ft}$, and initial sequents $F\bar{n} \to Ft$, respectively, in place of standard derivations.

This approach was not taken in this paper for the practical reason that it was found to complicate the treatment.

An approach on the same lines, which may however lead to a simpler treatment, has been suggested by A.S. Troelstra: namely, to take derivations as being built up, not from formulas, but from equivalence classes of formulas under the relation of computational equivalence.

9.3.2. Definition. An Ind inference is called *convertible* if the Ind term is closed.

9.3.3. Convention. If the proper variable a of an Ind inference does not actually occur in the formula Fa, we will assume, for convenience, that the Ind term is 0.

9.4. *Redundant variables in* $\mathcal{S}$(H) and $\mathcal{N}$(H)

9.4.1. Definition. A variable in a derivation in $\mathcal{S}$(H) or $\mathcal{N}$(H) is *redundant* if it is (free and) not used anywhere in that derivation as a proper variable.

9.4.2. Given a derivation in $\mathcal{S}$(H) or $\mathcal{N}$(H), we can effectively transform it into one without redundant variables, by replacing all redundant variables in it by (say) 0.

Of course, this may result in changing the end-sequent (in $\mathcal{S}$) or the assumptions and conclusion (in $\mathcal{N}$) if these contain any redundant variables. Also (more important for our purposes) it may change a non-convertible Ind inference into a convertible one.

Further, observe the following:

(i) $\mathcal{D}$ has no redundant variables iff $\varphi\mathcal{D}$ has no redundant variables.

(ii) If $\mathcal{D}_1$ has no redundant variables and $\mathcal{D}_1$ conv $\mathcal{D}_2$ then $\mathcal{D}_2$ has no redundant variables.

If Π_1 has no redundant variables and Π_1 conv Π_2 then Π_2 has no redundant variables.

(Note that (ii) applies not only to the conversions in the negative fragment that we have considered so far, but also to the conversions of § 13 for the full systems.)

So we can assume from now on (if we wish) that all derivations we deal with have no redundant variables (as well as the PVP: see § 2.5). (Of course, if a derivation has no redundant variables, it does not follow that all its subderivations have no redundant variables: think of the case where it ends with a $\forall$R (or $\forall$I), or Ind.)

§ 10. Strong equivalence in $\mathcal{S}^-$(H) *

10.1. *Statement of result*

Strong equivalence in $\mathcal{S}^-$(H), $\equiv$, is defined as in 4.1.2. We can again prove:

Theorem 1. $\mathcal{D}_1 \equiv \mathcal{D}_2 \Longleftrightarrow \varphi\mathcal{D}_1 = \varphi\mathcal{D}_2$.

10.2. *Some notes on the proof of Theorem* 1

10.2.1. Lemma 4.4(b) must be replaced by:

(b′) *If* $\mathcal{D}$ *is* TCF *and* $\varphi\mathcal{D}$ *is* $\dfrac{(Q),(R)}{P}$, *then* $\mathcal{D}$ *is* $(Q),(R) \to P$.

10.2.2. Consider the proof of Lemma 4.7. The case $l\mathcal{D}_1 = 0$ or $l\mathcal{D}_2 = 0$ works as before (using Lemma 4.4(b′) now).

Now suppose $l\mathcal{D}_1 > 0$ and $l\mathcal{D}_2 > 0$. There are four new cases: $r\mathcal{D}_1 = \text{Ind}$, and $r\mathcal{D}_2 = c\text{L}$ *or* Cut *or* Contr *or* Ind. These are treated as before.

Further, case (10): "cL, c'L" must be modified. Subcases (i) and (ii), namely, $r\Pi$ is an I-rule or E-rule respectively, are treated as before. But we must consider two further subcases (in place of (iii) there):

Subcase (iii′). $r\Pi = \text{M}$. Like (iii), but using now Lemma 4.4(b′).

Subcase (iv). $r\Pi = \text{Ind}$. Then by the proof of Proposition 8.5.2, there is a TCF $\mathcal{D}$ such that $\varphi\mathcal{D} = \Pi$ and $r\mathcal{D} = \text{Ind}$. So, by one of the new cases above ("Ind, cL"), $\mathcal{D}_1 \equiv \mathcal{D} \equiv \mathcal{D}_2$.

* This section should be compared with § 4.

§ 11. Essential conversions in $\mathcal{S}^-(\mathrm{H})$ and conversions in $\mathcal{N}^-(\mathrm{H})$ *

11.1. *Definitions*

11.1.1. (a) The *equational part* E of a derivation $\mathcal{D}$ is that part of $\mathcal{D}$ defined inductively as follows.

(i) Every MIS of $\mathcal{D}$ is in E;

(ii) if both US's of a cut are in E, then so is the LS.

(b) An *equational* (eqnl) *cut* of $\mathcal{D}$ is a cut in the equational part of $\mathcal{D}$.

(c) $\mathcal{D}$ is an *equational derivation* if its equational part is the whole of $\mathcal{D}$.

Note. The concepts of equational part of a derivation and equational cut can be thought of as generalizations of $\perp$-part and $\perp$-cut respectively (see §5.13).

11.1.2. (a) An *Ind-equational* (Ind-eqnl) *system* of cuts in a derivation $\mathcal{D}$ is a sequence of cuts in $\mathcal{D}$ having the form:

$$\dfrac{\begin{matrix}\mathcal{G}_1 \\ \Gamma_1 \to A_1\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{G}_m \\ \Gamma_m \to A_m\end{matrix} \qquad (Q_1),(R_1) \to P_1 \quad \cdots \quad (Q_n),(R_n) \to P_n}{\vdots \qquad\qquad (m+n-1 \text{ cuts})}$$

($m \geqslant 1, n \geqslant 1$), where $r\mathcal{G}_i = \text{Ind}$ ($1 \leqslant i \leqslant m$) (see 8.3.4(b)); $(Q_j),(R_j) \to P_j$ are MIS's ($1 \leqslant j \leqslant n$); and for each of these $m+n-1$ cuts, the cut formula in the RUS is (a descendant of) one of the MI formulas Q_j, R_j (i.e., *not* a formula occurrence in one of the Γ_i), and the cut formula in the LUS is (a descendant of) one of the Ind formulas A_i or MI formulas P_j. (Hence all the cut formulas are among the Q_j, R_j and P_j, and all the Ind formulas A_i are among the Q_j and R_j.)

Note. (i) We have not defined "descendant" (of a formula or indexed formula occurrence), but the meaning should be clear.

(ii) There is an abuse of notation here: the subscripts j in $(Q_j),(R_j) \to P_j$ are (of course) not indices.

* This section should be compared with §5.

(b) Each cut in an Ind-equational system is called an *Ind-equational cut*.

11.1.3. An *Ind-principal cut* in a derivation is one in which the cut formula is an Ind formula in the LUS, and a principal formula in the RUS.

11.1.4. A cut is *permissible* if it is equational, Ind-equational or Ind-principal.

11.1.5. (a) A derivation in $\mathcal{S}^-(\mathrm{H})$ is *normal* if:

(i) it has no cuts apart from permissible cuts, and

(ii) it has no convertible Ind inference.

(b) A derivation in $\mathcal{N}^-(\mathrm{H})$ is *normal* if no conversion rules apply to it; in other words:

(i) it has no maximal formulas, and

(ii) it has no convertible Ind inference.

11.1.6. A derivation in $\mathcal{S}^-(\mathrm{H})$ is *strictly normal* if it is normal and contr-normal (5.3.2).

11.1.7. We define, as in 5.1.2, the relations

(a) $\mathcal{D}_1 \succ \mathcal{D}_2$,

(b) $\Pi_1 \succ \Pi_2$,

(c) $\mathcal{D} \geq \mathcal{D}'$,

(d) $\Pi \succ\!\succ \Pi'$,

(e) $\mathcal{D}_1 \succ \mathcal{D}_2$ with cut formula A,

(f) $\Pi_1 \succ \Pi_2$ with maximal formula A.

Further (for Ind-conversions):

(g) "$\mathcal{D}_1 \succ \mathcal{D}_2$ with Ind triple (Fa, a, t)" (or, more loosely: "with Ind formula Ft") means: a is the proper variable, t the Ind term, and Ft the Ind formulation of the inference involved.

(h) "$\Pi_1 \succ \Pi_2$ with Ind triple (Fa, a, t)" (or: "with Ind formula Ft"): similarly.

11.2. *Statements of results*

The main results in this section are:

Theorem 1. $\mathcal{D}_1 \succ \mathcal{D}_2 \Rightarrow \varphi\mathcal{D}_1 \succ\!\succ \varphi\mathcal{D}_2$.

Theorem 2. $\mathcal{D} \mapsto \Pi_1 \succ \Pi_2 \Rightarrow \exists \mathcal{D}_1, \mathcal{D}_2\ (\mathcal{D} \geqslant \mathcal{D}_1 \succ \mathcal{D}_2 \mapsto \Pi_2)$.

Theorem 3. $\mathcal{D}$ *normal* $\Rightarrow \varphi\mathcal{D}$ *normal.*

Theorem 4. $\varphi\mathcal{D}$ *normal* $\Rightarrow \exists \mathcal{D}_0\ (\mathcal{D} \geq \mathcal{D}_0$ *and* $\mathcal{D}_0$ *normal*).

Note. These are the same as in 5.1.3, except that here, in Theorems 3 and 4, "cut-free" is replaced by "normal".

11.3. *Some remarks on our definition of normality in* $\mathcal{S}^-$(H)

11.3.1. The point of our definition of permissible cut, and hence of normality in $\mathcal{S}^-$(H), can perhaps best be appreciated by an examination of the proofs of Theorems 3 and 4 below (or rather, the lemma for Theorem 4: see §11.8).

What about *strict normality*? It follows from Proposition 11.9 below that all our theorems hold with the concept of normality (in $\mathcal{S}^-$(H)) replaced throughout by strict normality. (Compare with §6.9.) Moreover, strict normality has the following *stability* property with respect to conversions.

11.3.2. **Proposition.** *If* $\mathcal{D}$ *is strictly normal and* $\mathcal{D}$ conv $\mathcal{D}'$, *then*

(a) $\mathcal{D}'$ *is strictly normal.*

(b) $\mathcal{D}'$ conv $\mathcal{D}$.

Proof. If $\mathcal{D}$ is strictly normal, then (as can be seen) the only conversions possible in $\mathcal{D}$ are the following permutative conversions:

(1) permuting two equational cuts,

(2) permuting two Ind-equational cuts (in the same Ind-eqnl system),

(3) permuting two contractions.

The result follows from this.

11.3.3. However, this stability property does not hold for the concept of normality in $\mathcal{S}^-$(H), i.e., Proposition 11.3.2 is false with "normal" in place of "strictly normal", as will be seen from the proof of Proposition 11.9 below (case (2(c): $r\mathcal{D}$ = 'ad-principal cut).

11.4. *Proof of Theorem* 1

We now turn to the proofs of Theorems 1–4 in the present case.

Theorem 1 is proved as before (§5.2). There is an extra case to prove, namely that if $\mathcal{D}_1 \succ \mathcal{D}_2$ by an Ind-convn$_i$ ($i = 1, 2, 3$) then $\varphi\mathcal{D}_1 \succ\!\succ \varphi\mathcal{D}_2$ by a sequence of Ind-convns$_i$ (with the same Ind triple).

11.5. *Lemmas for Theorem* 2

Before considering the proof of Theorem 2, we point out that all the lemmas for Theorem 2 of §5 (i.e., §§5.4–5.7) still hold. Consider, in particular (cf. §5.5):

11.5.1. Lemma. $\forall\mathcal{D}\ \exists\mathcal{D}'$ s.t. $\mathcal{D} \geq \mathcal{D}'$ *(by contr convns) and* $\mathcal{D}'$ *is contr-normal.*

Proof. We extend the proof in §5.5 (by induction on $l\mathcal{D}$) with the new case:

Case 9. $r\mathcal{D}$ = Ind. Say

$$\mathcal{D} = \frac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to F0\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ (Fa_\alpha), \Delta \to Fa^+\end{matrix}}{\Gamma, \Delta \to Ft}\ .$$

Suppose $\alpha \neq \emptyset$. (For $\alpha = \emptyset$, the proof is simpler.) By induction hypothesis $\exists$ contr-normal $\mathcal{D}_i'$ ($i = 1, 2$) s.t. $\mathcal{D}_i \geq \mathcal{D}_i'$ (by contr convns), where

$$\mathcal{D}_1' = \frac{\dfrac{\begin{matrix}\mathcal{E}_1 \\ \Gamma' \to F0\end{matrix}}{\vdots}}{\Gamma \to F0}\ \text{(Contrs)}, \qquad \mathcal{D}_2' = \frac{\dfrac{\begin{matrix}\mathcal{E}_2 \\ Fa_{\sigma_1}, \ldots, Fa_{\sigma_n}, \Delta' \to Fa^+\end{matrix}}{\vdots}}{Fa_\alpha, \quad \Delta \to Fa^+}\ \text{(Contrs)},$$

with Γ', Δ' singular, and $\alpha = \{\sigma_1, \ldots, \sigma_n\}$. Put

$$\mathcal{D}' = \dfrac{\overset{\mathcal{E}_1}{\Gamma' \to F0} \qquad \dfrac{\dfrac{\overset{\mathcal{E}_2}{Fa_{\sigma_1}, \ldots, Fa_{\sigma_n}, \Delta' \to Fa^+}}{\vdots}}{Fa_\alpha, \quad \Delta' \to Fa^+}\ (\text{Contrs of } Fa_{\sigma_1}, \ldots, Fa_{\sigma_n}).}{\dfrac{\dfrac{\Gamma', \Delta' \to Ft}{\vdots}\ (\text{Contrs})}{\Gamma, \Delta \to Ft}}\ (\text{Ind})$$

Then $\mathcal{D}'$ is contr-normal and $\mathcal{D} \geqslant \mathcal{D}'$ (by contr convns).

11.6

Theorem 2. $\mathcal{D} \mapsto \Pi_1 \succ \Pi_2 \Rightarrow \exists\, \mathcal{D}_1, \mathcal{D}_2\ (\mathcal{D} \geq \mathcal{D}_1 \succ \mathcal{D}_2 \mapsto \Pi_2)$.

(Further: (a) If $\Pi_1 \succ \Pi_2$ with maximal formula G, then $\mathcal{D}_1 \succ \mathcal{D}_2$ with cut formula G, and (b) if $\Pi_1 \succ \Pi_2$ with Ind formula Gt then $\mathcal{D}_1 \succ \mathcal{D}_2$ with Ind formula Gt.)

Proof. (a) If $\Pi_1 \succ \Pi_2$ with maximal formula G (i.e., by a logical conversion), then the proof is as in §5.8.

(b) Suppose now $\Pi_1 \succ \Pi_2$ with Ind formula Gt. We show $\exists \mathcal{D}_1, \mathcal{D}_2$ s.t. $\mathcal{D} \geq \mathcal{D}_1$ and $\mathcal{D}_1 \succ \mathcal{D}_2$ with Ind formula Gt, and $\varphi\mathcal{D}_2 = \Pi_2$. The proof is by induction on $l\mathcal{D}$, with cases according to $r\mathcal{D}$. It goes as for part (a), except for the case $r\mathcal{D} = \text{Ind}$:

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}'}{\Gamma \to F0} \qquad \overset{\mathcal{D}''}{(Fa_\alpha), \Delta \to Fa^+}}{\Gamma, \Delta \to Fs} \quad \mapsto \quad \dfrac{\begin{matrix}\Gamma \\ \varphi\mathcal{D}' \\ F0\end{matrix} \qquad \begin{matrix}[Fa_\alpha], \Delta \\ \varphi\mathcal{D}'' \\ Fa^+\end{matrix}}{Fs} = \Pi_1 .$$

Let I be the Ind inference involved in the conversion of Π_1. There are three subcases.

Subcases (i) *and* (ii). I is in $\varphi\mathcal{D}'$ or $\varphi\mathcal{D}''$. As for part (a).

Subcase (iii). I is the Ind shown (as $r\Pi_1$): so Fs is Gt. Then define $\mathcal{D}_1$ and $\mathcal{D}_2$ by: $\mathcal{D}_1 = \mathcal{D}$, and $\mathcal{D}_1 \succ \mathcal{D}_2$ with Ind formula Fs.

11.7

Theorem 3. *$\mathcal{D}$ normal $\Rightarrow \varphi\mathcal{D}$ normal.*

Proof. As in §5.10. (The point is that permissible cuts in $\mathcal{D}$ cannot give rise to maximal formulas in $\varphi\mathcal{D}$, as is easily seen.)

11.8

Theorem 4. *$\varphi\mathcal{D}$ normal $\Rightarrow \exists\mathcal{D}_0$ ($\mathcal{D} \geq \mathcal{D}_0$ and $\mathcal{D}_0$ normal).*

Proof. This is proved just as before (§5.12), except that the proof of Lemma 5.11, on which it depends, changes. We re-state this lemma and prove it.

Lemma (cf. §5.11). *Suppose*

$$\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ A_\alpha, \Delta \to B\end{matrix}}{\Gamma_{\times\alpha}, \Delta \to B}\ \text{(Cut)}$$

with $\mathcal{D}_1$ and $\mathcal{D}_2$ normal and $\varphi\mathcal{D}$ normal. Then $\exists\mathcal{D}^0$ s.t. $\mathcal{D} \geq \mathcal{D}^0$ and $\mathcal{D}^0$ is normal.

Proof. This proceeds, as in §5.11, by induction on $(\bar{\alpha}, l\mathcal{D})$. (We will omit the index "α" below, since the index associated with the cut formula A will be α in all cases below.) However there are some complications now.

Firstly, there are, in addition to cases 1–4 (cf. §5.11) three more cases to consider, corresponding to the three new types of cut listed in 8.3.6:

Case 5. A is MI in both US's;

Case 6. A is Ind in LUS and MI in RUS;

Case 7. A is Ind in LUS and principal in RUS.

But in each of these cases, the cut shown (as $r\mathcal{D}$) is *permissible*: equational in case 5, Ind-equational in case 6 and Ind-principal in case 7. Hence $\mathcal{D}$ is normal. So just take $\mathcal{D}^0 = \mathcal{D}$.

Secondly (and this is where things become complicated) we must re-examine case 2 (A passive in one US). The proof in § 5.11 proceeded on the assumption that $r\mathcal{D}_1$ and $r\mathcal{D}_2$ were *not Cut* (since $\mathcal{D}_1$ and $\mathcal{D}_2$ were cut-free there).

So if A is passive in $\mathcal{D}_1$ (or $\mathcal{D}_2$) with $r\mathcal{D}_1$ (or $r\mathcal{D}_2$, respectively) not cut, we can proceed as in case 2 in § 5.11.

However we must now also consider some new cases (subcases of case 2):

(i) $r\mathcal{D}_1 = r\mathcal{D}_2 =$ Cut;

(ii) $r\mathcal{D}_1 =$ Cut, and A in $\mathcal{D}_2$ is MI *or* principal;

(iii) $r\mathcal{D}_2 =$ Cut, and A in $\mathcal{D}_1$ is MI *or* principal *or* Ind.

(Any other subcases of case 2 can be treated by the methods of § 5.11.)

Note that if $r\mathcal{D}_1$ or $r\mathcal{D}_2$ is Cut, it must be *permissible* (since $\mathcal{D}_1$ and $\mathcal{D}_2$ are normal). So these new cases can be conveniently subdivided in the following way:

(a) $r\mathcal{D}_2 =$ M *or* eqnl cut, $r\mathcal{D}_1 =$ M *or* eqnl cut.

(b) $r\mathcal{D}_2 =$ M *or* eqnl cut, $r\mathcal{D}_1 =$ Ind *or* Ind-eqnl cut.

(c) $r\mathcal{D}_2 =$ Ind-eqnl cut, $r\mathcal{D}_1 =$ M *or* eqnl cut *or* Ind *or* Ind-eqnl cut, *or* A in $\mathcal{D}_1$ is principal.

(d) $r\mathcal{D}_2 =$ Ind-principal cut.

(c) $r\mathcal{D}_1 =$ Ind-principal cut.

These cases overlap with one another to some extent, but they include all the new subcases of case 2 (i.e., (i)–(iii) above) (as well as cases 5 and 6 again!). (Some combinations are impossible, e.g. "$r\mathcal{D}_1 =$ eqnl cut, A in $\mathcal{D}_2$ is principal", since this would require A to be both atomic and non-atomic.)

We will consider each of these five cases in turn.

(a) $r\mathcal{D}_2 =$ M *or* eqnl cut, $r\mathcal{D}_1 =$ M *or* eqnl cut.
Then $r\mathcal{D}$ is an eqnl cut. So take $\mathcal{D}^0 = \mathcal{D}$.

(b) $r\mathcal{D}_2 =$ M *or* eqnl cut, $r\mathcal{D}_1 =$ Ind *or* Ind-eqnl cut.
Then $r\mathcal{D}$ is an Ind-eqnl cut. So, again, take $\mathcal{D}^0 = \mathcal{D}$.

(c) $r\mathcal{D}_2 =$ Ind-eqnl cut, $r\mathcal{D}_1 =$ M *or* eqnl cut *or* Ind *or* Ind-eqnl cut, *or* A in $\mathcal{D}_1$ is principal.

Say

$$\mathcal{D}_2 = \dfrac{\dfrac{\cdots \quad \mathcal{G}_i \quad \cdots \quad (Q_j),(R_j) \to P_j \quad \cdots}{\vdots}}{A, \Delta \to B} \quad \begin{array}{l}(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)\\ \text{(Ind-eqnl cuts)}\end{array}$$

where

$$\mathcal{G}_i = \dfrac{\begin{matrix}\mathcal{D}_i' \\ \Delta_i' \to F_i 0\end{matrix} \quad \begin{matrix}\mathcal{D}_i'' \\ (F_i a_i), \Delta_i'' \to F_i a_i^+\end{matrix}}{\Delta_i', \Delta_i'' \to F_i t_i} \ (\text{Ind}) .$$

There are three subcases:

(c)-(i) A is in Δ_i' (for some i): say $\Delta_i' = A, \Delta^-$. Let

$$\mathcal{D}_3 = \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to A\end{matrix} \quad \begin{matrix}\mathcal{D}_i' \\ A, \Delta^- \to F_i 0\end{matrix}}{\Gamma, \Delta^- \to F_i 0} \ (\text{Cut}) .$$

By induction hypothesis (since $l\mathcal{D}_3 < l\mathcal{D}$ and $\varphi\mathcal{D}_3$, being a subderivation of $\varphi\mathcal{D}$, is normal) $\exists$ normal $\mathcal{D}_3^0$ s.t. $\mathcal{D}_3 \geq \mathcal{D}_3^0$. So take

$$\mathcal{D}^0 = \dfrac{\dfrac{\mathcal{G}_1 \quad \cdots \quad \dfrac{\begin{matrix}\mathcal{D}_3^0 \\ \Gamma, \Delta^- \to F_i 0\end{matrix} \quad \begin{matrix}\mathcal{D}_i'' \\ (F_i a_i), \Delta_i'' \to F_i a_i^+\end{matrix}}{\Gamma, \Delta^-, \Delta_i'' \to Ft} \quad \cdots \quad \mathcal{G}_m \quad \cdots \quad (Q_j),(R_j) \to P_j \quad \cdots}{\vdots \qquad \text{(Ind-eqnl cuts)}}}{\Gamma, \Delta \to B}$$

Then $\mathcal{D}^0$ is normal and $\mathcal{D} \geq \mathcal{D}^0$.

(c)-(ii) A is in Δ_i'' (for some i): say $\Delta_i'' = A, \Delta^-$. Let

$$\mathcal{D}_3 = \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to A\end{matrix} \quad \begin{matrix}\mathcal{D}_i'' \\ A, (F_i a_i), \Delta^- \to F_i a_i^+\end{matrix}}{\Gamma, (F_i a_i), \Delta^- \to F_i a_i^+} \ (\text{Cut}) .$$

By induction hypothesis (since $l\mathcal{D}_3 < l\mathcal{D}$ and $\varphi\mathcal{D}_3$, being a subderivation of $\varphi\mathcal{D}$, is normal) $\exists$ normal $\mathcal{D}_3^1$ s.t. $\mathcal{D}_3 \geq \mathcal{D}_3^0$. So take

$$\mathcal{D}^0 = \dfrac{\dfrac{\mathcal{I}_1 \quad \cdots \quad \dfrac{\dfrac{\mathcal{D}_i'}{\Delta_i' \to F_i 0} \quad \dfrac{\mathcal{D}_3^0}{(F_i a_i), \Gamma, \Delta^- \to F_i a_i^+}}{\Gamma, \Delta_i', \Delta^- \to F_i t_i} \quad \cdots \quad \mathcal{I}_m \quad \cdots \quad (Q_j), (R_j) \to P_j \quad \cdots}{\vdots} \text{(Ind-eqnl cuts)}}{\Gamma, \Delta \to B}$$

(c)-(iii) A is a descendant of a Q_j or R_j. Then A in $\mathcal{D}_1$ cannot be principal. So, since $r\mathcal{D}_1$ = M *or* eqnl cut *or* Ind *or* Ind-eqnl cut, it can be seen that $r\mathcal{D}$ is itself an Ind-eqnl cut!

To make this clearer: suppose, e.g., that $r\mathcal{D}_1$ is an Ind-eqnl cut; say

$$\mathcal{D}_1 = \dfrac{\dfrac{\cdots \quad \mathcal{I}_k' \quad \cdots \quad (Q_l'), (R_l') \to P_l' \quad \cdots}{\vdots} \text{(Ind-eqnl cuts)}}{\Gamma \to A} \qquad (1 \leqslant k \leqslant p,\ 1 \leqslant l \leqslant q)$$

Then in fact

$$\mathcal{D} = \dfrac{\dfrac{\cdots \quad \mathcal{I}_k' \quad \cdots \quad \mathcal{I}_i \quad \cdots \quad (Q_l'), (R_l') \to P_l' \quad \cdots \quad (Q_j), (R_j) \to P_j \quad \cdots}{\vdots} \text{(Ind-eqnl cuts)}}{\Gamma, \Delta \to B}$$

$(1 \leqslant k \leqslant p,\ 1 \leqslant i \leqslant m,\ 1 \leqslant l \leqslant q,\ 1 \leqslant j \leqslant n)$, where the cuts shown form a single Ind-equational system.

So take $\mathcal{D}^0 = \mathcal{D}$.

(Similarly if $r\mathcal{D}_1$ = M *or* eqnl cut *or* Ind-eqnl cut.)

(d) $r\mathcal{D}_2$ = Ind-principal cut. Say

$$\mathcal{D}_2 = \dfrac{\dfrac{\dfrac{\mathcal{D}_2'}{\Delta' \to F0} \quad \dfrac{\mathcal{D}_2''}{(Fa), \Delta'' \to Fa^+}}{\Delta', \Delta'' \to Ft} \quad \dfrac{\mathcal{D}_2'''}{Ft, \Delta''' \to B}}{A, \Delta \to B} \text{(Cut)}$$

where Ft in $\mathcal{D}_2'''$ is principal (and $A, \Delta = \Delta', \Delta'', \Delta'''$).

Again there are three subcases, according as A is in Δ', Δ'' or Δ''':

(d)-(i) A is in Δ': say $\Delta' = A, \Delta^-$. (So $\Delta = \Delta^-, \Delta'', \Delta'''$.) Let

$$\mathcal{D}_3 = \frac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to A\end{matrix} \quad \begin{matrix}\mathcal{D}_2' \\ A, \Delta^- \to F0\end{matrix}}{\Gamma, \Delta^- \to F0}\ \text{(Cut)}\ .$$

By induction hypothesis, $\exists$ normal $\mathcal{D}_3^0$ s.t. $\mathcal{D}_3 \geq \mathcal{D}_3^0$. So take

$$\mathcal{D}^0 = \frac{\dfrac{\begin{matrix}\mathcal{D}_3^0 \\ \Gamma, \Delta^- \to F0\end{matrix} \quad \begin{matrix}\mathcal{D}_2'' \\ (Fa), \Delta'' \to Fa^+\end{matrix}}{\Gamma, \Delta^-, \Delta'' \to Ft} \qquad \begin{matrix}\mathcal{D}_2''' \\ Ft, \Delta''' \to B\end{matrix}}{\Gamma, \Delta \to B}\ \text{(Cut)}\ .$$

The cut shown here is again Ind-principal, so $\mathcal{D}^0$ is normal.

(d)-(ii) A is in Δ'': similarly.

(d)-(iii) A is in Δ''': say $\Delta''' = A, \Delta^-$. (So $\Delta = \Delta', \Delta'', \Delta^-$.)
Suppose for simplicity that $r\mathcal{D}_2''' = \wedge\mathrm{L}$ or $\forall\mathrm{L}$:

$$\mathcal{D}_2''' = \frac{\begin{matrix}\mathcal{D}_2'''' \\ G, A, \Delta^- \to B\end{matrix}}{Ft, A, \Delta^- \to B}\ (\wedge\mathrm{L}\ \text{or}\ \forall\mathrm{L})\ .$$

Let

$$\mathcal{D}_3 = \frac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to A\end{matrix} \quad \begin{matrix}\mathcal{D}_2'''' \\ A, G, \Delta^- \to B\end{matrix}}{\Gamma, G, \Delta^- \to B}\ \text{(Cut)}$$

By induction hypothesis, $\exists$ normal $\mathcal{D}_3^0$ s.t. $\mathcal{D}_3 \geq \mathcal{D}_3^0$. So take

$$\mathcal{D}^0 = \frac{\dfrac{\mathcal{D}_2' \quad \mathcal{D}_2''}{\Delta', \Delta'' \to Ft} \qquad \dfrac{\begin{matrix}\mathcal{D}_3^0 \\ G, \Gamma, \Delta^- \to B\end{matrix}}{Ft, \Gamma, \Delta^- \to B}\ \begin{matrix}(\wedge\mathrm{L}\ \text{or}\ \forall\mathrm{L}) \\ \text{(Cut)}\end{matrix}}{\Gamma, \Delta \to B}\ .$$

The cut shown here is again Ind-principal, so $\mathcal{D}^0$ is normal. (Similarly if $r\mathcal{D}_2''' = \supset$L.)

(e) $r\,\mathcal{D}_1$ = Ind-principal cut. Say

$$\mathcal{D}_1 = \dfrac{\begin{matrix}\mathcal{G}\\ \Gamma^+ \to C\end{matrix} \qquad \begin{matrix}\mathcal{D}_1'\\ C, \Gamma' \to A\end{matrix}}{\Gamma \to A}\ \text{(Cut)}$$

where C in $\mathcal{D}_1'$ is principal (and $\Gamma = \Gamma^+, \Gamma'$).

Suppose, e.g., that $r\mathcal{D}_1' = \supset$L:

$$\mathcal{D}_1' = \dfrac{\begin{matrix}\mathcal{D}_1''\\ \Gamma'' \to D\end{matrix} \qquad \begin{matrix}\mathcal{D}_1'''\\ E, \Gamma''' \to A\end{matrix}}{C, \Gamma' \to A}\ (\supset\text{L})$$

(so $C = D \supset E$, and $\Gamma' = \Gamma'', \Gamma'''$). Let

$$\mathcal{D}_3 = \dfrac{\begin{matrix}\mathcal{D}_1'''\\ E, \Gamma''' \to A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ A, \Delta \to B\end{matrix}}{E, \Gamma''', \Delta \to B}\ \text{(Cut)}$$

By induction hypothesis, $\exists$ normal $\mathcal{D}_3^0$ s.t. $\mathcal{D}_3 \preceq \mathcal{D}_3^0$. So take

$$\mathcal{D}^0 = \dfrac{\begin{matrix}\mathcal{G}\\ \Gamma^+ \to C\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_1''\\ \Gamma'' \to D\end{matrix} \qquad \begin{matrix}\mathcal{D}_3^0\\ E, \Gamma''', \Delta \to B\end{matrix}}{C, \Gamma', \Delta \to B}\ (\supset\text{L})}{\Gamma, \Delta \to B}\ \text{(Cut)}$$

The cut shown here is again Ind-principal, so $\mathcal{D}^0$ is normal. (Similarly, if $r\mathcal{D}_1' = \wedge$L or $\forall$L.)

11.9. *Normality and strict normality*

We conclude this section with the following proposition, in view of which all our theorems hold when the concept of normality (in $\mathcal{S}^-$(H)) is replaced throughout by strict normality (11.1.6; cf. §6.9).

Proposition. *If $\mathcal{D}$ is normal, then $\exists \mathcal{D}^+$ s.t. $\mathcal{D} \geq \mathcal{D}^+$ and $\mathcal{D}^+$ is strictly normal.*

Proof. Let $\mathcal{D}'$ be the contr-normal derivation obtained from $\mathcal{D}$ by the proof of Lemma 11.5.1 (there and in §5.5), i.e., by repeated contraction conversions. Then $\mathcal{D}'$ is *not* necessarily normal (see case 2(c) below), but we will define a $\mathcal{D}^+$ such that $\mathcal{D}' \geq \mathcal{D}^+$ and $\mathcal{D}^+$ is strictly normal.

More precisely, if

$$\mathcal{D}' = \dfrac{\begin{array}{c}\mathcal{E}\\ \Gamma' \to A\end{array}}{\vdots} \quad \text{(Contrs)}$$

with Γ' singular, then $\mathcal{D}^+$ will have the form

$$\dfrac{\begin{array}{c}\mathcal{E}^+\\ \Gamma' \to A\end{array}}{\vdots} \quad \text{(Contrs)}$$

with $\mathcal{E} \geq \mathcal{E}^+$ (and $\mathcal{E}^+$ also strictly normal).

The proof, i.e., construction of ($\mathcal{E}^+$, and hence) $\mathcal{D}^+$ from $\mathcal{D}'$, is by induction on $l\varphi\mathcal{D}$ ($= l\varphi\mathcal{D}'$). There are two cases:

Case 1. $r\mathcal{D} \neq$ Cut. This is straightforward. Suppose e.g. that $r\mathcal{D} = \supset$R. (Other subcases are similar.) So

$$\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, (A_\alpha) \to B\end{array}}{\Gamma \to A \supset B}$$

($\alpha \neq \emptyset$ say). Then (cf. §5.5, case 3) $\mathcal{D}_1 \geq$ contr-normal $\mathcal{D}_1'$, where

$$\mathcal{D}_1' = \dfrac{\dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma', A_{\sigma_1}, \ldots, A_{\sigma_n} \to B\end{array}}{\vdots}}{\Gamma, \quad A_\alpha \quad \to B} \quad \text{(Contrs)}$$

with Γ' singular, and $\alpha = \{\sigma_1, ..., \sigma_n\}$; and furthermore,

$$\mathcal{D}' = \left.\begin{array}{ll} \dfrac{\dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma', A_{\sigma_1}, ..., A_{\sigma_n} \to B\end{array}}{\begin{array}{c}\vdots\end{array}} \quad (\text{Contr } A_{\sigma_1}, ..., A_{\sigma_n})}{\dfrac{\Gamma', \quad A_\alpha \quad \to B}{\Gamma' \to A \supset B}} & \end{array}\right\} \mathcal{E}$$

$$\dfrac{\begin{array}{c}\vdots\end{array} \quad (\text{Contrs})}{\Gamma \to A \supset B}$$

By induction hypothesis (applied to $\mathcal{D}_1$), $\exists$ strictly normal $\mathcal{E}_1^+$ s.t. $\mathcal{E}_1 \geq \mathcal{E}_1^+$.

So define ($\mathcal{E}^+$, and hence) $\mathcal{D}^+$, simply by replacing $\mathcal{E}_1$ by $\mathcal{E}_1^+$ in ($\mathcal{E}$, respectively) $\mathcal{D}$.

Case 2. $r\mathcal{D}$ = Cut. Since $\mathcal{D}$ is normal, this is a permissible cut. So there are three subcases:

Case 2(a). $r\mathcal{D}$ = eqnl cut. But then $\mathcal{D}$ is equational (11.1.1 (c)), hence $\mathcal{D}' = \mathcal{D}$. So take $\mathcal{D}^+ = \mathcal{D}$.

Case 2(b). $r\mathcal{D}$ = Ind-eqnl cut:

$$\mathcal{D} = \dfrac{\dfrac{\begin{array}{c}\mathcal{G}_i\\ ... \;\; \Gamma_i \to A_i \;\; ... \;\; (Q_j), (R_j) \to P_j \;\; ...\end{array}}{\vdots}}{\Gamma_1, ..., \Gamma_m, \Delta \to A} \quad \begin{array}{l}(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)\\ \\ (\text{Ind-eqnl cuts})\end{array}$$

Then (by the method of 11.5.1) $\mathcal{G}_i \geq$ contr-normal $\mathcal{G}_i'$, where

$$\mathcal{G}_i' = \dfrac{\dfrac{\begin{array}{c}\mathcal{E}_i\\ \Gamma_i' \to A_i\end{array}}{\vdots} \quad (\text{Contrs})}{\Gamma_i \to A_i},$$

with $r\,\mathcal{E}_i$ = Ind, and Γ_i' singular $(1 \leqslant i \leqslant m)$; and furthermore,

$$\mathcal{D}' = \cfrac{\cfrac{\cfrac{\cdots \quad \overset{\mathcal{E}_i}{\Gamma_i' \to A_i} \quad \cdots \quad (Q_j),(R_j) \to P_j \quad \cdots}{\vdots}\ (m+n-1 \text{ cuts})}{\Gamma_1', \ldots, \Gamma_m', \Delta \to A}}{\cfrac{\vdots}{\Gamma_1, \ldots, \Gamma_m, \Delta \to A}\ (\text{Contrs})} \ .$$

(Δ is singular, since it consists of descendants of the Q_j, R_j.)

By induction hypothesis (applied to the $\mathcal{G}_i$), $\mathcal{E}_i \geq$ strictly normal $\mathcal{E}_i^+$ $(1 \leqslant i \leqslant m)$. Further, note that the cuts shown in $\mathcal{D}'$ still form an Ind-eqnl system!

So we define $\mathcal{D}^+$ simply by replacing $\mathcal{E}_i$ by $\mathcal{E}_i^+$ $(1 \leqslant i \leqslant m)$ in $\mathcal{D}'$.

Case 2(c). $r\mathcal{D}$ = Ind-principal cut. Take e.g. the subcase:

$$\mathcal{D} = \cfrac{\overset{\mathcal{D}_1}{\Gamma \to A \wedge B} \qquad \cfrac{\overset{\mathcal{D}_2}{A_\alpha, \Delta \to C}}{A \wedge B_\alpha, \Delta \to C}\,(\wedge\text{L})}{\Gamma_{\times\alpha}, \Delta \to C}$$

where $r\,\mathcal{D}_1$ = Ind. Then (again as in 11.5.1, and §5.5, case 6) $\mathcal{D}_i \geq$ contr-normal $\mathcal{D}_i'$ $(i = 1, 2)$ where

$$\mathcal{D}_1' = \cfrac{\overset{\mathcal{E}_1}{\Gamma' \to A \wedge B}}{\cfrac{\vdots}{\Gamma \to A \wedge B}}\ (\text{Contrs})\,, \qquad \mathcal{D}_2' = \cfrac{\cfrac{\overset{\mathcal{E}_2}{A_{\sigma_1}, \ldots, A_{\sigma_n}, \Delta' \to C}}{\vdots}\ (n\wedge\text{L's})}{\cfrac{A \wedge B_{\sigma_1}, \ldots, A \wedge B_{\sigma_n}, \Delta' \to C}{\cfrac{\vdots}{A \wedge B_\alpha, \quad \Delta \to C}\ (\text{Contrs})}}$$

with $r\,\mathcal{E}_1$ = Ind, Γ' and Δ' singular, and $\alpha = \{\sigma_1, \ldots, \sigma_n\}$; and furthermore (cf. § 5.5, case 1):

$$\mathcal{D}' = \dfrac{\dfrac{\dfrac{\overbrace{\begin{matrix}\mathcal{E}_1 & & \mathcal{E}_1 \\ \Gamma' \to A \wedge B & \ldots & \Gamma' \to A \wedge B\end{matrix}}^{n \text{ copies}} \qquad \dfrac{\begin{matrix}\mathcal{E}_2 \\ \dfrac{A_{\sigma_1}, \ldots, A_{\sigma_n}, \Delta' \to C}{\vdots}\end{matrix}}{A \wedge B_{\sigma_1}, \ldots, A \wedge B_{\sigma_n}, \Delta' \to C}\,(n \wedge \text{L's})}{\vdots}\,(n \text{ cuts: } A \wedge B_{\sigma_1}, \ldots, A \wedge B_{\sigma_n})}{\dfrac{\Gamma'_{\times\sigma_1}, \ldots, \Gamma'_{\times\sigma_n}, \Delta' \to C}{\dfrac{\vdots}{\Gamma_{\times\alpha}, \quad \Delta \to C}}\,(\text{Contrs})}$$

But (for $n > 1$) these n cuts are *not all permissible*!

So we must now permute these cuts up past the $\wedge$L inferences, to obtain a $\wedge$L follows by a cut, repeated n times; i.e., we define:

$$\mathcal{D}^+ = \begin{array}{c} \dfrac{\begin{matrix}\mathcal{E}_1^+ \\ \Gamma' \to A \wedge B\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{E}_2^+ \\ A_{\sigma_1}, A_{\sigma_2}, \ldots, A_{\sigma_n}, \Delta' \to C\end{matrix}}{A \wedge B_{\sigma_1}, A_{\sigma_2}, \ldots, A_{\sigma_n}, \Delta' \to C}\,(\wedge\text{L})}{\Gamma'_{\times\sigma_1}, A_{\sigma_2}, \ldots, A_{\sigma_n}, \Delta' \to C}\,(\text{Cut } A \wedge B_{\sigma_1}) \\ \vdots \\ \dfrac{\dfrac{\begin{matrix}\mathcal{E}_1^+ \\ \Gamma' \to A \wedge B\end{matrix} \qquad \dfrac{\overline{\Gamma'_{\times\sigma_1}, \ldots, \Gamma'_{\times\sigma_{n-1}}, \quad A_{\sigma_n}, \quad \Delta' \to C}}{\Gamma'_{\times\sigma_1}, \ldots, \Gamma'_{\times\sigma_{n-1}}, \; A \wedge B_{\sigma_n}, \quad \Delta' \to C}\,(\wedge\text{L})}{\Gamma'_{\times\sigma_1}, \ldots, \Gamma'_{\times\sigma_n}, \Delta' \to C}\,(\text{Cut } A \wedge B_{\sigma_n})}{\dfrac{\vdots}{\Gamma_{\times\alpha}, \quad \Delta \to C}}\,(\text{Contrs}) \end{array}$$

where $\mathcal{E}_i \geq$ strictly normal $\mathcal{E}_i^+$ $(i = 1, 2)$, by the induction hypothesis applied to $\mathcal{D}_i$. (We assume w.l.o.g. that the n $\wedge$L's shown in $\mathcal{D}_2'$ are applied to $A_{\sigma_1}, \ldots, A_{\sigma_n}$ in that order.)

Then $\mathcal{D}^+$ is strictly normal.

The other subcases (where the cut formula in the RUS of $r\mathcal{D}$ is the principal formula of an $\supset$L or $\forall$L) are treated similarly.

§12. Reduction sequences in $\mathcal{S}^-(\mathrm{H})$ and $\mathcal{N}^-(\mathrm{H})$ *

12.1. *Normalization and strong normalization*

12.1.1. *Reduction sequences* in $\mathcal{S}^-(\mathrm{H})$ and $\mathcal{N}^-(\mathrm{H})$, and *proper reduction sequences* in $\mathcal{S}^-(\mathrm{H})$, are defined as in 6.1.1 (d), (e) and 6.4.1 (c).

The *normalization theorem* for $\mathcal{S}^-(\mathrm{H})$ is the statement: "$\forall \mathcal{D}\ \exists \mathcal{D}'$ s.t. $\mathcal{D}$ red $\mathcal{D}'$ and $\mathcal{D}'$ is normal".

The *strong normalization theorem* for $\mathcal{S}^-(\mathrm{H})$ is the statement: "every proper reduction sequence in $\mathcal{S}^-(\mathrm{H})$ is finite" (cf. 6.8.1).

12.1.2. Now all the results of §6 (in particular Theorems 1–3) go through, with "cut-free" and "[strong] cut-elimination" replaced throughout by "normal" and "[strong] normalization".

12.1.3. **Corollary.** *The strong normalization theorem for* $\mathcal{S}^-(\mathrm{H})$ *is true.*

Proof. By the strong normalization theorem for $\mathcal{N}^-(\mathrm{H})$ (in fact for $\mathcal{N}(\mathrm{H})$: see Troelstra [23, ch. IV]).

12.1.4. **Note.** (a) Although the induction conversion rule in [23, ch.IV] is different from ours, the proof of strong normalization there still works for our system, if we take (in the notation of [23, 4.1.16]) the "induction value" of a derivation to be just δ, and notice that all standard derivations are "strongly valid under substitution" (proved by induction on the complexity of the conclusion).

(b) *Uniqueness of normal form in* $\mathcal{N}(\mathrm{H})$ is also proved in [23, ch. IV].

12.2. *Induction-free normal forms*

We will now define a class of derivations in $\mathcal{S}^-(\mathrm{H})$, whose normal forms contain no cuts (apart from equational cuts) or induction, in terms of the syntactic structure of their end-sequents; and hence, from this, define a class of derivations in $\mathcal{N}^-(\mathrm{H})$ whose normal forms contain no induction (Theorem 1 below, 12.2.5).

* This section should be compared with §6.

12.2.1. Definitions. (a) The set of subformulas of a formula A can be partitioned into two subsets: $\mathrm{p}(A)$ and $\mathrm{n}(A)$, the *positive* and *negative* subformulas of A. These are defined simultaneously by induction on the complexity of A.

(i) If A is atomic: $\mathrm{p}(A) = \{A\}$, $\mathrm{n}(A) = \emptyset$.

(ii) If A is $B \supset C$: $\mathrm{p}(A) = \{A\} \cup \mathrm{p}(C) \cup \mathrm{n}(B)$, $\mathrm{n}(A) = \mathrm{n}(C) \cup \mathrm{p}(B)$.

(iii) If A is $B \wedge C$ or $B \vee C$: $\mathrm{p}(A) = \{A\} \cup \mathrm{p}(B) \cup \mathrm{p}(C)$,
$\mathrm{n}(A) = \mathrm{n}(B) \cup \mathrm{n}(C)$.

(iv) If A is $\forall x\, Fx$ or $\exists x\, Fx$: $\mathrm{p}(A) = \{A\} \cup \mathrm{p}(Fa)$, $\mathrm{n}(A) = \mathrm{n}(Fa)$.

(Note: although we only consider applications to the negative fragment below, we have given the definition for the full language, since later (§ 13) it will be used for the full system.)

(b) A logical constant *occurs positively* in A if it occurs in a formula of $\mathrm{p}(A)$; it *occurs negatively* in A if it occurs in a formula of $\mathrm{n}(A)$.

(c) A logical constant *occurs positively* [*negatively*] in a sequent $\Gamma \to A$ if it occurs positively [negatively] in A, or negatively [positively] in some formula of Γ.

(d) For any set $c_1, \ldots, c_m, c_{m+1}, \ldots, c_n$ of logical constants, a sequent is said to be in the class $\{c_1^+, \ldots, c_m^+, c_{m+1}^-, \ldots, c_n^-\}$ if *none* of $c_1, \ldots, c_m$ occur *positively* in it, and *none* of $c_{m+1}, \ldots, c_n$ occur *negatively* in it.

(This notation has been used by V.P. Orevkov.)

(e) A derivation

$$\begin{array}{c} \mathcal{D} \\ \Gamma \to A \end{array} \qquad \text{or} \qquad \begin{array}{c} \Gamma \\ \Pi \\ A \end{array}$$

is in the class $\{c_1^+, \ldots, c_{m+1}^-, \ldots\}$ if the sequent $\Gamma \to A$ is.

(f) A sequent is *closed* if all the formulas in it are closed, i.e., contain no free variables.

12.2.2. Lemma. *If* $\mathcal{D}$ *is in* $\{c_1^+, \ldots, c_{m+1}^-, \ldots\}$, *then so is* [*are*] *its immediate subderivation*[*s*], *unless* $r\mathcal{D} =$ Cut *or* Ind *with non-atomic cut or induction formula.*

Proof. By inspection.

12.2.3. Lemma. *Suppose* $\mathcal{D}$ *is normal, in the class* $\{\forall^+\}$ *and without* *redundant variables* (§9.4). *Then:*

(a) $r\mathcal{D} \neq \forall\mathrm{R}$,

(b) $r\mathcal{D} \neq \mathrm{Ind}$,

(c) *if* $r\mathcal{D} = \mathrm{Cut}$, *then it is an equational cut, with closed cut formula.*

Proof. (a) $r\mathcal{D} = \forall\mathrm{R} \Rightarrow \mathcal{D}$ would not be in $\{\forall^+\}$.

(b) Suppose $r\mathcal{D} = \mathrm{Ind}$, with Ind term t. Then:

(i) t closed $\Rightarrow$ this Ind would be convertible $\Rightarrow$ $\mathcal{D}$ not normal:

(ii) t not closed $\Rightarrow$ $\mathcal{D}$ would have a redundant variable [1].

(c) Suppose

$$\mathcal{D} = \frac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \to A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ A, \Delta \to B\end{matrix}}{\Gamma, \Delta \to B}\ (\mathrm{Cut})$$

This cut must be permissible. But $r\mathcal{D}_1 \neq \mathrm{Ind}$ (by the same argument as in (b)), so it must be an equational cut. Also A must be closed, since otherwise $\mathcal{D}$ would have a redundant variable.

12.2.4. Lemma. *If* $\mathcal{D}$ *is normal, in* $\{\forall^+\}$ *and without redundant variables, then so is* [*are*] *its immediate subderivation*[*s*].

Proof. (a) Suppose

$$\mathcal{D} = \frac{\mathcal{D}_1}{\Gamma \to A}\ .$$

It is immediate that $\mathcal{D}_1$ is normal.

$\mathcal{D}_1$ is in $\{\forall^+\}$, by 12.2.2. It has no redundant variables, since $r\mathcal{D} \neq \forall\mathrm{R}$ (by 12.2.3(a)).

(b) Suppose

$$\mathcal{D} = \frac{\mathcal{D}_1 \qquad \mathcal{D}_2}{\Gamma \to A}\ .$$

It is immediate that $\mathcal{D}_1$ and $\mathcal{D}_2$ are normal.

[1] Since ($\varphi\mathcal{D}$, and hence) $\mathcal{D}$ is assumed to have PVP (2.5.5), a free variable in t could not also be a proper variable in $\mathcal{D}$.

$\mathcal{D}_1$ and $\mathcal{D}_2$ are in $\{\forall^+\}$, by 12.2.2 and 12.2.3(b), (c). They have no redundant variables since $r\mathcal{D} \neq$ Ind (12.2.3(b)), and if $r\mathcal{D} =$ Cut, then the cut formula is closed (12.2.3(c)).

12.2.5. Theorem 1. (a) *If $\mathcal{D}$ is in $\{\forall^+\}$ and has a closed end-sequent, then any normal form of $\mathcal{D}$ contains no* Ind, *and no cuts, apart from equational cuts.*

(b) *If Π is in $\{\forall^+\}$ and has closed assumptions and conclusion, then its normal form contains no* Ind.

Proof. (a) From 12.2.4 and 12.2.3 (b), (c). We assume that the normalization procedure includes transforming the derivation to one without redundant variables. This will not affect the end-sequent, since it is closed.

(b) follows immediately from (a).

12.2.6. Remark. This theorem is for the negative fragment only. We extend part (b) later (13.6.1) to the full system $\mathcal{N}$(H).

§13. Normalization by Hinata's method

13.1. *Introduction*

We will describe a method for proving normalizability of derivations in $\mathcal{N}(\mathrm{H})$ (i.e. with full logic) by (quantifier-free) induction on ϵ_0. The idea is due to Hinata [6] who used it to prove normalizability of the terms of Gödel's theory T.

Briefly, the method is as follows. Let Π be a derivation in $\mathcal{N}(\mathrm{H})$. Now choose any derivation $\mathcal{D}_0$ in $\mathcal{S}(\mathrm{H})$ such that $\varphi\mathcal{D}_0 = \Pi$. Then let

$$S: \qquad \mathcal{D}_0 \succ_G \mathcal{D}_1 \succ_G \mathcal{D}_2 \succ_G \cdots$$

be a reduction sequence from $\mathcal{D}_0$ ($\succ_G$ for "Gentzen-type conversion") obtained by the method of Gentzen [5], as adapted by Scarpellini [18] for intuitionistic systems. This induces a reduction sequence

$$\varphi S: \qquad \varphi\mathcal{D}_0 \succ\!\succ \varphi\mathcal{D}_1 \succ\!\succ \varphi\mathcal{D}_2 \succ\!\succ \cdots$$

from Π. Moreover, if $o(\mathcal{D})$ is the ordinal assigned to a derivation $\mathcal{D}$ by the method of Gentzen [5], then

$$o(\mathcal{D}_0) > o(\mathcal{D}_1) > o(\mathcal{D}_2) > \ldots ;$$

so S must terminate, and hence so must φS, and in normal form.

Although this method is simple, it works (apparently) only for derivations Π in the class $\{\forall^+, \exists^-, \vee^-\}$ (12.2.1).

We will now go through this more carefully.

13.2. *Hinata's work*

Hinata [6] proved normalizability of the terms of Gödel's theory T by the following method.

With each term t is associated (non-uniquely) a "tree of terms" θ showing how t is built up, and having t itself at the bottom node. Then a conversion relation on trees of terms is defined, so that if θ_1 conv θ_2 and t_i is at the bottom node of θ_i ($i = 1, 2$), then t_1 conv t_2.

Now if we consider the well-known "isomorphism" between terms and natural deduction derivations, due to Curry, Howard and others (see Troelstra [23, 4.1.6]), then (roughly), as terms correspond to natural deduction derivations, so trees of terms correspond to sequent calculus derivations.

Hinata's method then proceeds as in § 13.1, but in the language of terms rather than derivations.

13.3. *G-conversions in* $\mathcal{S}$(H)

13.3.1. Let $\mathcal{D}$ be a derivation in $\mathcal{S}$(H), with end-sequent consisting of closed atomic formulas. We will define the notion of *G*-conversion of $\mathcal{D}$: $\mathcal{D} \succ_G \mathcal{D}'$, using the method of Gentzen [5], adapted by Scarpellini [18] for intuitionistic systems.

The *end-piece* of $\mathcal{D}$ is defined as in [18], viz. it contains those sequents below which there are no inferences other than cuts and contractions.

We assume $\mathcal{D}$ has no redundant variables (§ 9.4), so of the "preparatory steps" considered by Gentzen [5] or Scarpellini [18], we only have to worry about the removal from the end-piece of logical initial sequents (which all occur as upper sequents of trivial cuts).

Next, the following three possibilities arise:

(1) There is an Ind with lower sequent in the end-piece. Since $\mathcal{D}$ has no redundant variables, its end-piece contains no free variables at all, and hence any such Ind is convertible. So we perform an Ind-conversion (as in § 9.2(b)) on such an inference.

If there is no such Ind:

(2) we perform a "logical reduction" on $\mathcal{D}$, involving a cut in the end-piece and one of the logical constants.[2] As described by Scarpellini [18], this is done in two steps:

(a) a logical reduction at this cut, as in Gentzen [5], which results in an "almost intuitionistic" derivation; and

(b) a pruning of this, to obtain an intuitionistic derivation again.

Either (1) or (2) applies unless

(3) the end-piece of $\mathcal{D}$ is all of $\mathcal{D}$.

[2] This is analogous, but *not* identical, to the "logical conversions" of 9.1.1(a).

In cases (1) and (2) (calling the new derivation $\mathcal{D}'$) we write: $\mathcal{D} \succ_G \mathcal{D}'$.

13.3.2. Now the following hold.

(1) If $\mathcal{D} \succ_G \mathcal{D}'$ by an Ind-conversion, then $\varphi\mathcal{D} \succ\!\succ \varphi\mathcal{D}'$ by a sequence of Ind-conversions.

(2) If $\mathcal{D} \succ_G \mathcal{D}'$ by a "logical reduction" involving the logical constant c and cut formula A, then $\varphi\mathcal{D} \succ\!\succ \varphi\mathcal{D}'$ by a sequence of c-conversions, with maximal formula A. (Note that c may be $\vee$ or $\exists$ here.)

(3) If the end-piece of $\mathcal{D}$ is all of $\mathcal{D}$, then $\varphi\mathcal{D}$ is normal.

The proof of (1) is exactly as for the case of Ind-conversions in Theorem 1 of § 11 (see § 11.4). The proof of (2) is a refinement of that for c-conversions in the same theorem. (We omit details.) As for (3), just notice that $\mathcal{D}$ contains only atomic formulas.[3]

13.3.3. *Ordinal assignment.* We assign to each derivation $\mathcal{D}$ in $\mathcal{S}$(H) an ordinal $o(\mathcal{D})$ as in Gentzen [5], except for the case of the Ind inference (which is different from the *CJ*-inference, *op. cit.*):

$$\frac{\Gamma \to F0 \qquad (Fa_\alpha), \Delta \to Fa^+}{\Gamma, \Delta \to Ft}\ (\mathrm{Ind})\ .$$

If the upper sequents have ordinals $\omega^\alpha + \ldots$ and $\omega^\beta + \ldots$ (in Cantor normal form), then the *line of inference* has the ordinal $\omega^{\max(\alpha,\beta)+1}$.

The *degree* of a cut or Ind is defined, as in [5], as the degree of the cut or Ind formula (i.e., the total number of logical constants occurring in it).

Then, as in [5] and [18]:

$$\mathcal{D} \succ_G \mathcal{D}' \Rightarrow o(\mathcal{D}) > o(\mathcal{D}')\ .$$

(In the case of an Ind-conversion of $\mathcal{D}$, notice that the ordinal of the lower sequent of the standard derivation is finite, since the cuts in it all have degree less than that of the Ind formula.)

These considerations complete the argument sketched in § 13.1.

[3] The results of 13.3.2 were known to Scarpellini in 1972 (personal communication).

13.4. *The method for derivations in* $\{\forall^+, \exists^-, \vee^-\}$.

The method of § 13.3 applies to derivations

$$\begin{matrix}\Gamma\\ \Pi\\ A\end{matrix} \quad \left(\text{or } \overset{\mathcal{D}}{\Gamma \to A}\right)$$

where $\Gamma \cup \{A\}$ consists of closed atomic formulas. However it can be extended to the case that $\Gamma \to A$ is closed and in $\{\forall^+, \exists^-, \vee^-\}$. We briefly describe the method.

13.4.1. So suppose $\mathcal{D}$ is in $\{\forall^+, \exists^-, \vee^-\}$ and without redundant variables. We will define a derivation $\mathcal{D}^*$, by transfinite induction on $o(\mathcal{D})$, such that $\varphi\mathcal{D}^*$ is the normal form of $\varphi\mathcal{D}$.

If case (1) or (2) of 13.3.1 applies, define $\mathcal{D}^* = (\mathcal{D}')^*$ (where $\mathcal{D} \succ_G \mathcal{D}'$). If case (3) applies, let $\mathcal{D}^* = \mathcal{D}$.

Now suppose none of (1), (2) and (3) applies. Then, by Scarpellini [18, Theorem 2]:

(4) there is a logical inference in $\mathcal{D}$, with lower sequent in the end-piece, such that its principal formula has a descendant in the end-sequent.

We will call such an inference a *critical inference.*

Now choose one such inference, and consider cases, according to the inference rule. Notice that it cannot be $\forall$R, $\exists$L or $\vee$L, since $\mathcal{D}$ is in $\{\forall^+, \exists^-, \vee^-\}$. Suppose, e.g., it is $\supset$L:

$$\mathcal{D} = \begin{array}{c} \dfrac{\overset{\mathcal{D}_1}{\Gamma \to A} \qquad \overset{\mathcal{D}_2}{B_\beta, \Delta \to C}}{A \supset B_\beta, \Gamma_{\times\beta}, \Delta \to C}\,(\supset\text{L}) \\ \vdots \qquad \text{(Cuts, Contrs)} \\ \hline A \supset B_\alpha, \quad \theta \quad \to D \end{array}$$

where $A \supset B_\alpha$ is a descendant of $A \supset B_\beta$, formed (in general) by contracting (descendants of) $A \supset B_\beta$ with other indexed formulas.

Now by permuting the cuts shown in $\mathcal{D}$ (but not the contractions) above the $\supset$L, we obtain a derivation

$$\mathcal{D}' = \begin{array}{l} \dfrac{\begin{matrix}\mathcal{D}_1' & \mathcal{D}_2' \\ \Gamma' \to A & B_\beta, \Delta' \to C'\end{matrix}}{A \supset B_\beta, \Gamma'_{\chi\beta}, \Delta' \to C'}\,(\supset \mathrm{L}) \\ \qquad\qquad \vdots \qquad\qquad (\text{Contrs only}) \\ \overline{A \supset B_\alpha, \quad \theta \quad \to D} \end{array}$$

where

$$\mathcal{D}_1' = \begin{array}{c}\dfrac{\mathcal{D}_1}{\ } \\ \vdots\ (\text{Cuts})\end{array}, \qquad \mathcal{D}_2' = \begin{array}{c}\dfrac{\mathcal{D}_2}{\ } \\ \vdots\ (\text{Cuts})\end{array} \qquad \text{and} \qquad \varphi\mathcal{D}' = \varphi\mathcal{D}\,.$$

Now we cannot say that $o(\mathcal{D}') < o(\mathcal{D})$. However it is true that $o(\mathcal{D}_i') < o(\mathcal{D})$ $(i = 1, 2)$. Moreover, $\mathcal{D}_1'$ and $\mathcal{D}_2'$ are in $\{\forall^+, \exists^-, \vee^-\}$ (by 12.2.2) and have no redundant variables. So by the induction hypothesis applied to *these* derivations, we can construct $(\mathcal{D}_1')^*$ and $(\mathcal{D}_2')^*$ as required. Then $\mathcal{D}^*$ is defined by substituting $(\mathcal{D}_i')^*$ for $\mathcal{D}_i'$ $(i = 1, 2)$ in $\mathcal{D}'$.

The other cases for critical inferences ($\supset$R, $\wedge$R, $\wedge$L, $\vee$R, $\exists$R) are treated similarly.

The fact that $\varphi\mathcal{D}^*$ is the *normal form* of $\varphi\mathcal{D}$ is easily shown by induction on $o(\mathcal{D})$ (following the inductive construction of $\mathcal{D}^*$ from $\mathcal{D}$). In fact $\varphi\mathcal{D}^*$ is normal in the strong sense of having no maximal *segments* (Prawitz [17, II, 3.1.2–3]), even though the reduction of $\varphi\mathcal{D}$ to $\varphi\mathcal{D}^*$ given by this method does not include the permutative conversions for $\vee$E and $\exists$E (7.8.1).

13.4.2. The reasons for restricting derivations to the class $\{\forall^+, \exists^-, \vee^-\}$ are as follows.

Firstly, if the critical inference in case (4) were $\forall$R or $\exists$L, the subderivation obtained by the reduction (corresponding to $\mathcal{D}_1'$ or $\mathcal{D}_2'$ in the case $\supset$L, above) would, in general, have a redundant variable (namely the proper variable of this inference), and so there would be no guarantee now that (for case (1)) an Ind with lower sequent in the endpiece of this subderivation would be convertible.

Secondly, suppose we had a critical inference $\vee$L, and permuted it with the cuts below it to obtain a derivation $\mathcal{D}'$ (as with $\supset$L). Then, in general, $\varphi\mathcal{D}' \neq \varphi\mathcal{D}$! (This is the situation we encountered in §7.2: $\varphi\mathcal{D}$ reduces to $\varphi\mathcal{D}'$ by an improper reduction.)

13.5. *Remark on the type of reduction given by this method*

Considering again the correspondence between terms and derivations: the reductions given by this method need be neither "from the inside" ("strict reductions": see [23, 2.2.2]) nor "from the outside" (elimination of "main cuts", as in [14]). In fact (supposing s red s', t red t') we may have $(\lambda x \cdot s)t$ reducing to $s[x/t]$ *or* $(\lambda x \cdot s)t'$ *or* $(\lambda x \cdot s')t$ *unless* (in the last case) s red s' by the conversion of a subterm containing x. Further, the reductions are not necessarily "restricted" in the sense of [7], since we may have $\lambda x \cdot s$ red $\lambda x \cdot s'$, even (now) by the conversion of a subterm of s containing x.

Note that the above just states which conversions are possible at all; however, which conversions are actually available at any stage in a given reduction of a given term (or derivation in $\mathcal{N}(\mathrm{H})$) depends largely on which tree of terms (or derivation in $\mathcal{S}(\mathrm{H})$) was chosen to represent it at the start.

13.6. *Induction-free normal forms again*

This method yields the following simple corollary.

13.6.1. Corollary. *If* Π *is in* $\{\forall^+, \exists^-, \vee^-\}$ *and has closed assumptions and conclusion, then its normal form contains no* Ind.

Proof. Take any $\mathcal{D}$ such that $\varphi\mathcal{D} = \Pi$. Then the normal form of Π is $\varphi\mathcal{D}^*$, where $\mathcal{D}^*$ is obtained from $\mathcal{D}$ by the method of §13.4. It is easy to see that $\mathcal{D}^*$ (and hence $\varphi\mathcal{D}^*$) has no Ind (by induction on $o(\mathcal{D})$, following the construction of $\mathcal{D}^*$ from $\mathcal{D}$).

13.6.2. Remark. This is an extension of the result (§12, Theorem 1(b)) for $\mathcal{N}^-(\mathrm{H})$. However it is not the best possible, since D. Leivant (1974, unpublished) has shown, by a direct analysis of normal derivations in $\mathcal{N}(\mathrm{H})$, that the result actually holds for the class $\{\forall^+, \exists^-\}$.

13.7. *Assessment of Hinata's method of proving normalizability*

Since normalizability (in fact, strong normalizability) for $\mathcal{N}$(H) is known to hold (by a "computability" or "validity" type of argument: see [23, ch. IV]) we must ask what value, if any, the present proof has.

Firstly, it has independent interest, since it is a proof by induction on ϵ_0 (and, in fact, can be formalized in primitive recursive arithmetic plus quantifier-free induction on ϵ_0). Secondly, it gives information on the normal form obtained (§ 13.6). But its chief merit lies, I think, in its simplicity.

Its main drawback is that it works only for derivations in a limited class: $\{\forall^+, \exists^-, \vee^-\}$.

It would be interesting to see if this method could be refined to prove normalizability for arbitrary derivations in $\mathcal{N}$(H), or even strong normalizability.

Arndt [1] proves normalizability by an assignment of ordinals *directly* to derivations in $\mathcal{N}$(H). His method has the advantage of working for arbitrary derivations. However the method described here seems simpler.

It may also be possible to prove normalizability by a direct ordinal assignment to derivations in $\mathcal{N}$(H), similar to that of Howard [7] for the terms of Gödel's theory T (using again the isomorphism between terms and derivations).

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